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Aspects of proof in pupils' practice of school mathematics

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(translated by David Pimm)

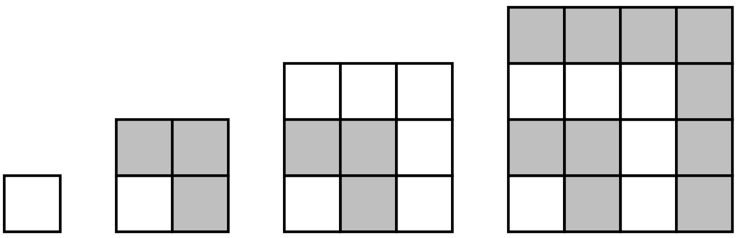
Introduction

This study is of the notion of proof from the point of view of the mathematical practices of pupils and not that of the logician. It relies upon an experimental approach which allows the processes of proof used in solving a problem to be seen more easily, in particular examining how the pupils arrive at their conviction of the validity of the proposed solution. In order to do this, we have used a social setting which requires spoken interaction (while minimising the intervention of the observer) which, by means of the discussion which occurs, allows the processes which underly the decisions taken by the pupils to become visible.

Levels and types of proof

Pragmatic versus conceptual proofs

The most elementary form of expression of a proof is one of direct showing. The operations and the concepts used are acts, neither differentiated nor articulated. This does not mean the complete absence of language, but it is not used as the fundamental means of transmitting knowledge. A classical example of a direct proof by showing is that of the result that the sum of the first n odd numbers is n^2.



These proofs rely on the ability of whoever sees the diagram to reconstruct the reasons that the prover has implicitly in mind but does not know how to explain otherwise (Sémadéni, 1984). In this type of proof, a central role is played by theorems-in-action (Vergnaud, 1981) which consist of those properties used by the solver in a problem, ones which she could not actually specify.

In this article, *pragmatic* proofs are those having recourse to actual action or showings, and by contrast, *conceptual* proofs are those which do not involve action and rest on formulations of the properties in question and relations between them. This detachment from action, from the here-and-now, does not happen by itself. It gets expressed in the language of the everyday. The action made explicit by this language carries traces of time order and duration, the person who acts and the context of her action. Nevertheless, use of such language already requires a certain distance so that the action can be described and made explicit. As Sémadéni suggests (thinking of an extension of action proofs), the movement to conceptual proofs lies essentially in taking account of the generic quality of those situations previously envisaged. The example below provides an illustration.

W $2 + 10 = 12$ $\quad 10 - 2 = 8$

$12 + 8 = 20$

Therefore $(2+10) + (10-2) = 20$

$(10+10) + (2-2) = 20$

It will always be 10 + 10

I've chosen 2 and it cancels out so if I choose another number between 1 and 10 it will always cancel out and will always be equal.

The move into conceptual proofs requires an altered position: the speaker must distance herself from the action and the processes of solution of the problem (recall that pupils when posed the question 'why' often reply by repeating the operations they have used to solve the problem). Knowledge, up until now acted out, becomes the object of reflection, discourse and indeed disagreements. The language of the everyday, whose main support is natural language, allows some movement in this direction. But there must be more than this to produce 'formal' proofs. Language must become a tool for logical deductions and not just a means of communication. The elaboration of this functional language requires in particular:

- a *decontextualisation,* giving up the actual object for the class of objects, independent of their particular circumstances;
- a *depersonalisation,* detaching the action from the one who acted and of whom it must be independent;

- a *detemporalisation*, disengaging the operations from their actual time and duration: this process is fundamental to the passage from the world of actions to that of relations and operations.

Among the various types of pragmatic and conceptual proofs, we have singled out four main types which hold a privileged position in the cognitive development of proof: naive empiricism, the crucial experiment, the generic example and the thought experiment. Actually, the former two types do not establish the truth of an assertion; we talk of *proof* because they are recognised as such by their producers. As we show below, there is a fundamental divide between the first two sorts of proof and the latter two. Moreover, for the generic example and the thought experiment, it is no longer a matter of 'showing' the result is true because 'it works'; rather, it concerns establishing the necessary nature of its truth by giving reasons. It involves a radical shift in the pupils' reasoning underlying these proofs.

Moreover, we claim that these forms of proof form a hierarchy, one which is taken into account in the order of presentation below. Where a particular type of proof falls in this hierarchy depends on how much the demands of generality and the conceptualisation of knowledge are involved. Thus moving from the generic example to the thought experiment, for instance, requires moving from action to internalised action, as well as involving a decontextualisation, which is one sign of a decisive change in the construction of knowledge.

Naive empiricism

Naive empiricism consists of asserting the truth of a result after verifying several cases. This very rudimentary (and as we know, insufficient) means of proving is one of the first forms of the process of generalisation (Piaget, 1978). But arising from a collection of problems posed to fifteen year olds, Bell (1979) found that 25 per cent of them based their answers only on the verification of a few cases. We can therefore expect naive empiricism to constitute a form resistant to generalisation.

The crucial experiment

The expression 'crucial experiment', coined by Francis Bacon (*Novum Organum*, 1620), refers to an experiment whose outcome allows a choice to be made between two hypotheses, it having been designed so that the outcome should be clearly different according to whether one or other hypothesis is the case. (Whether this experiment allows the rejection of one hypothesis or not, it does not allow us to assert that the other is true.)

We use the same expression for a slightly different process, one of verifying a proposition on an instance which 'doesn't come for free', asserting that 'if it works here, it will always work'. Here is an example from Bell (1976, cp. 10, p. 12): 'Jayne shows a complicated polygon, she can definitely say that the statement is true.' This type of validation is distinguishable from naive empiricism in that the pupil poses explicitly the problem of generality and resolves it by staking all on the outcome of a particular case that she recognises to be not too special.

The generic example

The generic example involves making explicit the reasons for the truth of an assertion by means of operations or transformations on an object that is not there in its own right, but as a characteristic representative of its class. The account involves the characteristic properties and structures of a class, while doing so in terms of the names and illustration of one of its representatives. Below is an example taken from Bezout (*Notes on Arithmetic*, 1832, p. 23):

> The remainder on dividing a number by 2 x 2 or 5 x 5 is the same as the remainder on dividing the number formed by the rightmost two digits by 2 x 20 or 5 x 5 . . .
>
> To fix these ideas, consider the number 43728 and the divisor 5 x 5. The number 43728 is equal to 43700+28. However, 43700 is divisible by 5 x 5, because 43700 is the product of 437 and 100, and as 100 is 10x10, or 5 x 2 x 5 x 2, the factor 100 is divisible by 5x5. The remainder on dividing 43728 by 5 x 5 or 25 is therefore the same as that on dividing 28 by 25.

The thought experiment

The thought experiment invokes action by internalising it and detaching itself from a particular representation. It is still coloured by an anecdotal temporal development, but the operations and foundational relations of the proof are indicated in some other way than by the result of their use, something which is the case for the generic example. (For more on the thought experiment, see Lakatos, 1976.) Here, for example, is the proof that Cauchy gave for the intermediate value theorem in his *Cours d'Analyse* of 1821:

> It suffices to show that the curve with equation $y = f(x)$ will cross the line $y = b$ at least once in the interval which includes the ordinates which correspond to the abscissae x_0 and X; however, it is clear that this will happen

> under the given hypotheses. Indeed, as the function f(x) is continuous between $x = x_0$ and $x = X$, the curve with equation $y = f(x)$ and which passes firstly through the point with coordinates $x0$, $f(x_0)$, and secondly through the point with coordinates X, f(X), will be continuous between those two points; and as the constant ordinate b of the line whose equation is $y = b$ lies between the ordinates, $f(x_0)$ and f(X), of the two points under consideration, the line necessarily passes between these two points, something it cannot do without meeting the above-mentioned curve in the interval.

An experimental study on types of proof

Presentation of the experimental setting

We chose for this study the problem of discovering and justifying a formula for the number of diagonals of any polygon. The 28 pupils aged thirteen and fourteen worked in pairs, though were only provided with one pen between two. The observer explained to them that

> you are to write a message which will be given to other pupils of your own age, which is to:
>
> > provide a means of calculating the number of diagonals of a polygon when you know the number of vertices it has.

(These last two lines were written on a sheet which was available to the pupils throughout the period of observation.)

The problem was given to the pupils without providing them with definitions of *polygon* and *diagonal*. This choice was made in order to allow the possibility of observing the (re-)construction of these concepts in the course of the solution, particularly with regard to refutations. In this way, we tried to place ourselves as close as possible to the focus of Lakatos' study (1976) which, along with the problem of proof, also examines that of the construction of mathematical knowledge.

The pupils were told that they had as much time as they wanted: they were to decide when they thought they had solved the problem. In fact, this activity formed the first part of the experiment. After a message had been offered by the pupils, the observer suggested difficulties that the recipients of the message could encounter in implementing their proposed method of calculation for certain polygons. In other words, in this second part, the observer offered counter- examples.

The experimental setting used thus consisted of two quite different parts, and the move from one to the other was achieved at the expense of breaking the 'experimental contract', that is to say involving the observer in a change of role from that of 'neutral observer' in order to offer counter-examples. For the majority of pupils, this breaking of the contract did not seem to have greatly affected their behaviour. It is true that the polygons the observer produced as counter-examples carried with them his authority, but this did not, for example, prevent pupils from rejecting these purported refutations.

Results of the problem solutions

The table below lists some of the wide range of solutions produced by the pupils during the first part of the experiment (about one hour). Eventually, more than half the pairs arrived at what we would call a correct solution. Only one pair, however, reached the classical formulation of the solution, f1(n) = n(n-3)12 in the first part of the experiment.

Pair	*First Part*
Christophe and Bertrand	$f_1(n) = n(n-3)/2$
Hamdi and Fabrice	n^2
Lionel and Laurent	n
Nadine and Elisabeth	$f(n+1) = f(n) + a(n+1)$ and $a(n+1) = a(n)+1$ where a(n) is the number of diagonals which should have been added to pass from P_{n-1} to P_n.
Martine and Laura	$f_2(n) = (n-3) + (n-3) + (n-4) + (n-5) + \ldots + 2 + 1$
Olivier and Stéphane	$f_2(n)$
Georges and Olivier	$f_2(n)$
Pierre and Philippe	n.s(n), where s(n) is the number of diagonals at a vertex
Lydie and Marie	n/2
Naïma and Valerie	n/2
Antoine and Damien	n/2
Evelyne and Christine	n/2 or n
Blandine and Elisabeth	n/2 or (n-1)/2
Pierre and Mathieu	n

These results support the a priori analysis made of the procedures and conceptions that the pupils would bring into play. Even the solution f(n) = 2n, which does not appear in the table, occurred: it was strongly defended by Christophe against Bertrand. (See the Appendix to this article on pp. 231-5.) The potency of the square as a model for diagonals explains the dominance of the solutions of the form $f(n) = n/2$.

The types of proof observed

We are interested here in the stages of validation of those results which appear, in the eyes of the pupils at least, to be conjectures. In addition, we are only considering the first part of the observations, where the observer has not yet intervened in the solution of the problem.

Pragmatic proofs

(a) Naive empiricism

We include here those conjectures which were taken from looking at a few cases; the question of their validity did not explicitly come up. But, on the other hand, the pupils showed, in their words or deeds, their confidence in these assertions. Rather than a characteristic of the pupils themselves, naive empiricism appears to be a state in which they found themselves, and they stayed there for reasons to do with the situation or their relationship with the knowledge itself.

The fact that particular pairs did not get past this level of naive empiricism can be described in terms of an obstacle, one whose source is the social interaction. Wanting to be right despite a disagreeing partner can get in the way of getting involved in problems of justification. Thus Christophe supported the conjecture $f(n)=2n$ (verified only in the case of a seven-sided polygon) against Bertrand's, without showing either uncertainty or a desire to look for a justification. (See the Appendix for more details.) Lydie defended her assertion that $f(n)=n/2$ against Marie every step of the way, by means of *ad hoc* arguments in the face of refutations. In these two cases, the pupils took advantage of their partner's difficulties to promote their own solution; they were not really involved in a collective effort to solve the problem.

Social interaction can thus provide an obstacle when pupils with very different conceptions are brought together. This was true in the case of Pierre and Mathieu, whose conceptions of polygon and diagonal were essentially distinct. The complexity involved in unifying their points of view and the emotional intensity of the conflict formed an obstacle to their getting involved in the question of proof. These problems arise, like those from the choice between two solutions, but the conflict is such that it favours one reading of the situation in terms of a game in which there is not much to lose. (In all that follows, P_n refers to an *n*-sided polygon).

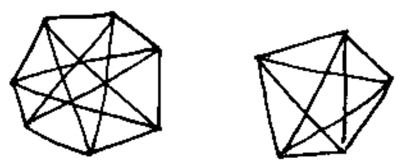
> After a very important discussion about what a diagonal actually is, and various proposals, the claim that $f(n)=n/2$ is arrived at from a consideration of P_4, P_6 and P_8: 'with a square, an eight-sided one and a six-sided one, so there you are, it must always be divided by two. The counter-example P_5 which Pierre comes up with is dealt with by an *ad hoc* adjustment to the claim: 'even, you divide by two . . . odd, you take one off it, which is confirmed by a laborious and

contentious check on P_7. The *ad hoc* nature of this adjustment confirms the naive empiricism which guides the two pupils' activities.

In fact, for Pierre, it is now a question of a 'theory' which Mathieu can refute if he does not agree; a conjecture about which he declares that he 'do[es] not really think that is it, but you always have to try rather than stay in the same place'. However, the pupils fail to undertake any other validatory steps.

For the figures whose number of vertices is even you divide by 2 the number of diagonals with respect to the number of vertices. For those figures whose number of vertices is odd, you take away 2 from the number of diagonals with respect to the number of vertices.

Without refuting it, Mathieu denies the previous conjecture that: 'the rule has to apply to every case. From looking at P_5 and P_7, as represented below,



he produces the conjecture f(n) = n, expressed in the following way:

To find the number of diagonal, it must agree with the number of vertices.

Here too it is a naive conjecture: 'I think that it is rather . . . that there are as many diagonals as vertices'.

So now the two conjectures are concurrent, 'gotta choose one of them'. Pierre suggests a way out of the situation: 'We'll write down each of our theories and we'll try to rule out those that are wrong'. This proposal is not followed, but it shows that naive empiricism is not intrinsic to these two pupils.

Instead of getting into a discussion about proof, the pupils refer it to the observer. Pierre would really like to have the solution and as for Mathieu: 'p'rhaps you think we've got it all wrong, or . . . or p'rhaps we've done it really well'.

Mathieu's fatalism and Pierre's impatience could comprise essential obstacles to their escaping from naive empiricism. To this can be added looking at the situation in terms of a riddle: 'it's a game . . . better to try', although one of little interest, of 'try anything you like'.

(b) The crucial experiment

The crucial experiment was observed as a means both of checking a result and as a weapon in discussions about validity between the two pupils from the same pair.

In every case, the crucial experiment was actually the checking of a proposition against a particular polygon with the deliberate intent of testing it. And thus, in the process, it involved the taking into account of generality. This goal is explicit in the work of Nadine and Elisabeth who, in this way, came up with a conjecture initially based on what could be called a primitive recursion.

From looking at P_4, P_5 and P_6 the pupils come up with the conjecture that $f(n+1) = f(n) + a(n+1)$ and $a(n+1) = a(n) + 1$ where $f(4) = 2$ and $a(4) = 2$. Before checking it, they predict for P_7: 'you'd have to add five to it usually' and 'usually, with 7, you'd have 14 diagonals, if that ever worked'.

The experiment holds for P_7 convex, but before that it had failed for P_7 concave: 'if it's like in the figure, you've not got the same number of diagonals [. . .] it must be without angles'. Also, at least for Nadine, this proof by experiment on P_7 is insufficient: 'Normally, if you're completely right, . . . with 8 sides you'd find six more of them.' She relies on the crucial experiment to decide: 'try it with 15 and then if it works for that, well then that means that it works for the others'. In fact, this experiment is carried out on P_{10}, because P_{15} seems too complex from the outset: '. . . 10 sides you should find 35 diagonals'. The experiment they have in mind confirms this result.

(c) The generic example

In three cases generic examples were used to establish the truth of the proposition $f2(n) = (n-3) + (n-3) + (n-4) +. . . + 2 + 1$. But the proof thus obtained never carries the conviction of both pupils, even if an apparent collaboration has united them. It is, in fact, in each of these cases, a crucial experiment which overcomes the doubts of the sceptical partner. The difficulty of these proofs lies in the fact that the speakers need to be agreed on the generic character of the example used, and thus that they share the same conceptions of the objects in question: otherwise the explanation that is developed will appear to be crucially tied to a particular case. In this empirical context, having recourse to a crucial experiment appears to be legitimate as a tool in the debate about validity. Below we discuss the case of Georges and Olivier:

The construction of the sequence (n-3), (n-3), (n-4), . . ., 2, 1 comes in response to 'a problem' raised by Georges while he is exploring the proposition that $f(n)=n.s(n)$ [where $s(n)$ is the number of diagonals at each vertex]: 'they [the diagonals] are doubling up'.

The question of a proof depends centrally on that of the first term of f_2. With the generic support of P_6, the pupils establish that this first term will be of the form n-3:

'The two segments which are next to the point you consider can't have diagonals, because they are like that. So, it's those on this side [...] if you take one, there are already two next to it ruled out.'

Once this is established, the construction sequence of f_2 is correct for P_6.

> Having announced the number of diagonals from each vertex, this consists of verifying it on a diagram stop by stop. However, this proof (based on a generic example) does not convince Olivier. He only believes the evidence of a crucial experiment with P_{14}.

Moreover, in the case of f_2, a discussion about reasons would be particularly difficult, as the difficulties in formulating messages attest. In effect, they involve expressing an iteration, and consequently a means of representation which is very intricate in natural language or requiring a high level of abstraction to be formulated in a more formal language (which would reduce the complexity to a linguistic one). This complexity can constitute an obstacle to the evolution of the kinds of proof used towards a more advanced level.

Conceptual proofs

(d) The thought experiment

Before including the proof style of a pair in this category, we require that the justification, which forms the basis of the validation of the proposition, should rest on an analysis of the properties of the objects in question. These properties are no longer evidenced on instances, but formulated in their generality. The action is internalised, invoked in the discourse which makes the proof explicit.

The thought experiment really appears as a means of underpinning the proposed propositions, in an effort to explain them. It does not involve particular situations. Decontextualisation, as can be seen with Olivier and Stéphane, is the central process of generalisation. The ideas involved are essentially based on perceptual experience and action. The reasons offered are bound up with actions, which are eventually internalised on the objects.

Expression of a thought experiment involves complex cognitive and linguistic constructions: at the linguistic level the difficulty is in expressing the operations on an abstract object (a class of objects). This ability must be operational to allow the construction of a proof. One elementary process of decontextualisation is erasing any traces of the particular in a formulation. But, as can be seen in the case of Olivier and Stéphane, such a process does not necessarily preserve relations.

> The construction of a proof of the assertion below is the result of a long and deliberate search for reasons on s(n) = n - 3.
>
> In a polygon if you have x vertices the number of diagonals which go from one point will be x-3 because there is the point you are leaning from and the two points which join it to the polygon.
>
> The first explanation rests on the generic example P6: 'There are the two points where they are joined . . . that's already taken care of: there are three left, as it

should be.' It is expressed in the following manner.

In a polygone if there are 6 points there will automatically be 3 diagonals for each point because in the boundary of the polygon there are two points which join it on: conclusion there are 3 which are diagonals.

But the expression of an explanation using an example is rejected as a proof 'there we've done an example . . . you mustn't use examples . . . it must be in general'.

Olivier proposes to introduce letters instead of numbers: 'you put x points';

In a polygone if there are x points there are automatically y diagonals for each point because in a boundary of the polygon there are two points which join it on: conclusion there are 3 which are the diagonals.

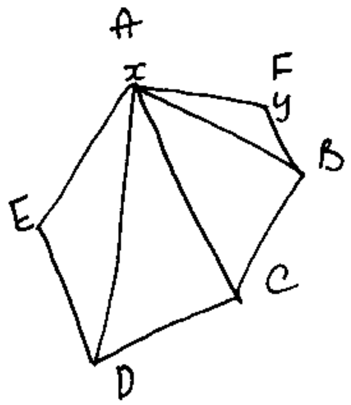
But replacing 6 by x and 3 by y fails to take into account the relation between the number of vertices and diagonals. The pupils then start looking in other directions. As Olivier suggests, 'there's not only one explanation'. The following attempt has the rhetorical style of geometry (see opposite). This explanation is, in fact, a formulation of a thought experiment.

Pupils must be able to express the properties of the objects concerned. The most common use ties the thought experiment to the generic example. It involves relying on individuals (particular cases), not as representatives of a class of objects, but as tools for linguistic expression: P6 traced by Stéphane for his geometric proof is not a generic example of polygon. It means polygon.

(e) Calculation on statements

We return here to proofs which have nothing to do with experience. They are intellectual constructions based on more-or-less formalised,

If the points are
not aligned



A which is adjacent to E and
F must have its diagonals
drawn to DCB which are the
points that are left.

more-or-less explicit theories of the ideas in question in the solution of the problem. These proofs appear as the result of an inferential calculation on statements. They rely on definitions or explicit characteristic properties.

We end by presenting below one of the two observed cases which reflect this type of proof, that of Antoine and Damien.

The claim asserted by the pupils is expressed in the following fashion:

To calculate the number of diagonals of a
polygon when you know the number of
vertices, you should divide the numbers
of vertices by two.

$$\frac{\text{vertices}}{2} = \text{no. of diagonals.}$$

It relies on definitions they adopted for polygon and diagonal: ‘the diagonals all pass through the same centre’ and ‘the sides must be parallel two by two’. These fundamentals came about when the problem of the domain of validity of dividing n by 2 came up: ‘and yeah, so, if it’s not an even number, it doesn’t work’ (Antoine). ‘It can’t be an odd number since we’ve said that the sides must be parallel in pairs’ (Damien).

Damien thinks that ‘It must be possible to prove it [. . .] by saying that each side, as it shares a diagonal and that each side touches another . . . you can easily show it.’

However, having gained conviction, this proof is not made explicit.

Conclusions

We have shown that the analysis of characteristics of linguistic expression of proofs is insufficient to make clear their level. In particular, the thought experiment, because it takes place in the language of the everyday using a primitive form of ‘proper names’ (for example, the name of a particular polygon), can seem to bear the hallmarks of a lower level of proof: generic example. In the end, it is knowledge about the process of production of the proof that allows a decision to be made about its effective validity and its level.

The hypothesis we had made about a hierarchy among the types of proofs has also been supported. But an analysis of the observed processes of validation allows us to go further and assert the existence of a break between naive empiricism and the crucial experiment on the one hand, and the generic example and the thought experiment on the other. This divide can be characterised as one of passing from a truth asserted on the basis of a statement of fact to one of an assertion based on reasons. It is a question of an actual change in the way of thinking about the problem, that is to say in the way of actually conceiving of and formulating the problem of the validity of an assertion. The source of this change can reside in a desire to get rid of an uncertainty, but a frequent obstacle in its effective realisation comes from the nature of the pupils’ conceptions of the mathematical ideas at stake, or even the linguistic means that they are able to construct or deploy.

We also recognise a connection between naive empiricism and the crucial experiment, when the latter is used at the end of a proof. Passage from the first to the second type of proof corresponds to taking into account the need to assure the generality of the supported conjecture. However, these two types hark back to the same empirical rationality (that is, one drawn from experience), according to which the accumulation of facts produces conviction in an assertion (Fischbein, 1982, p.17). But although naive empiricism disappears once conceptual proofs are brought into play, the crucial experiment can continue as an ultimate test to guarantee conviction, most noticeably when the assertion has been founded on a generic example. We find here an example of an operational cohabitation between empirical pragmatism and logical rationalism which Fischbein (op. cit.) suggests and which he considers as involving two types of rationality of practical validity differing in the degree to which empirical pragmatism retains its usefulness in everyday practice outside of mathematics. Thus, the crucial experiment means something different in social interaction, where it becomes a means of resolving completely a conflict over the validity of an assertion, or over the choice between two conjectures. We no longer consider it as a means of proof. except when it constitutes the refutation of an assertion. On the other hand, it brings a certain support to one of the two offered solutions, while not validating it entirely; its essential function is therefore to afford a defence against opposition.

Another connection is that between the generic example and the thought experiment. Passing from the former type of proof to the latter relies on a linguistic construction which involves a recognition and differentiation between the objects and relations involved in the solution of the problem; in other words a cognitive construction. A generic example constitutes a transitional stage in moving from pragmatic to conceptual proofs, in that it always requires a negotiation of the generic character of the example employed. This fragility encourages an evolution towards the thought experiment, which detaches itself from the particular.

Finally, we have made reference in our observations to proof, but it is only intended in principle to evoke the need to produce a 'mathematical' proof. The attempts of certain pupils to establish a proof by mathematical means runs up against the difficulty of providing them with a justification in the specific setting of the problem to be solved. The practice of proof requires reasoning and a specific state of knowledge at one and the same time. In addition, it involves a commitment to a problem-solving approach which is no longer one of effectiveness (a practical requirement) but one of rigour (a theoretical requirement).

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In the following pages, you will discover a cartoon. We invite you, while reading it, to enter the mathematical world of two French pupils aged 13-14 years. It indicates the way these pupils cope with a problem we have given them, how they come to an agreement about a common solution, and what arguments they use.

The scenario of this cartoon comes from analysis of the observed behaviour of those pupils while they worked together at the solution of the following problem:

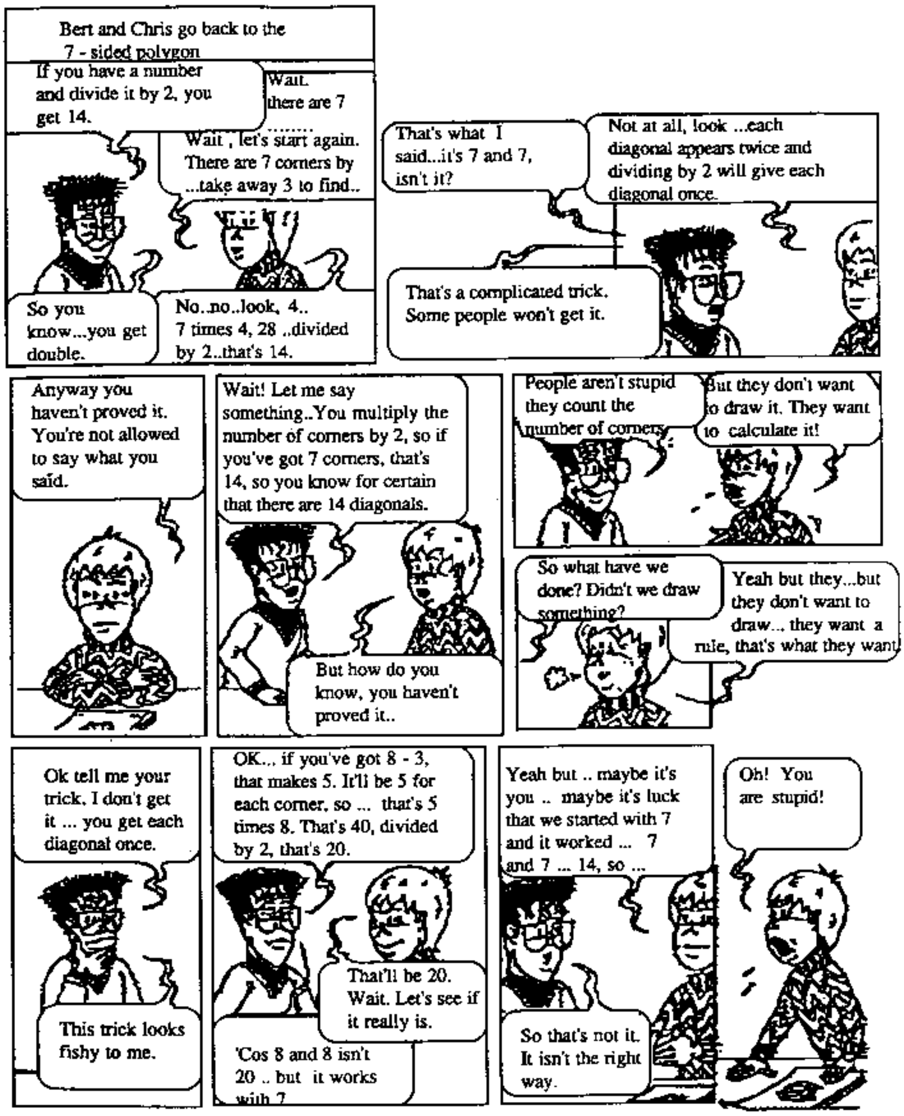
> *provide a means of calculating the number of diagonals of a polygon when you know the number of vertices it has.*

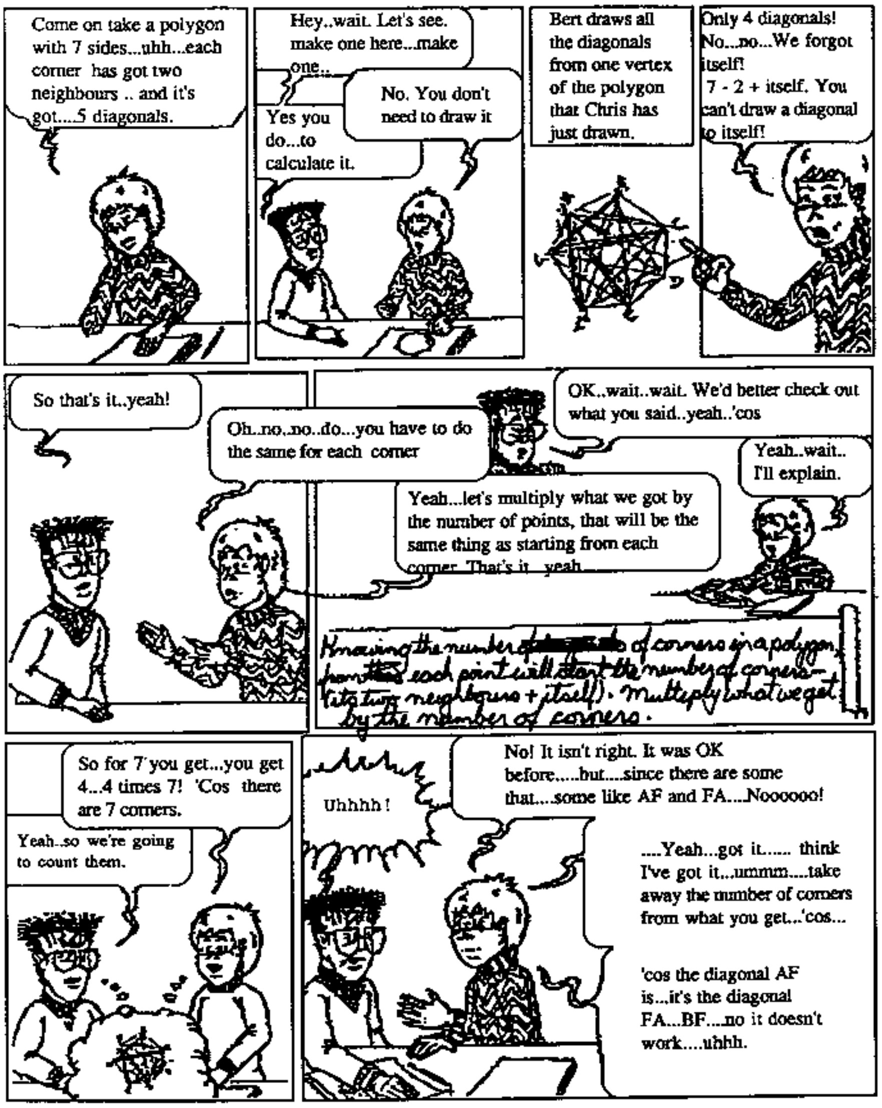
The answer to this task was to be expressed in a message to be addressed to, and to be used by, other 13 -14 year-old pupils.

Neither pupils' actions nor conversations are fictitious: we have chosen to express them by means of a cartoon to take their dynamics into account as far as possible.

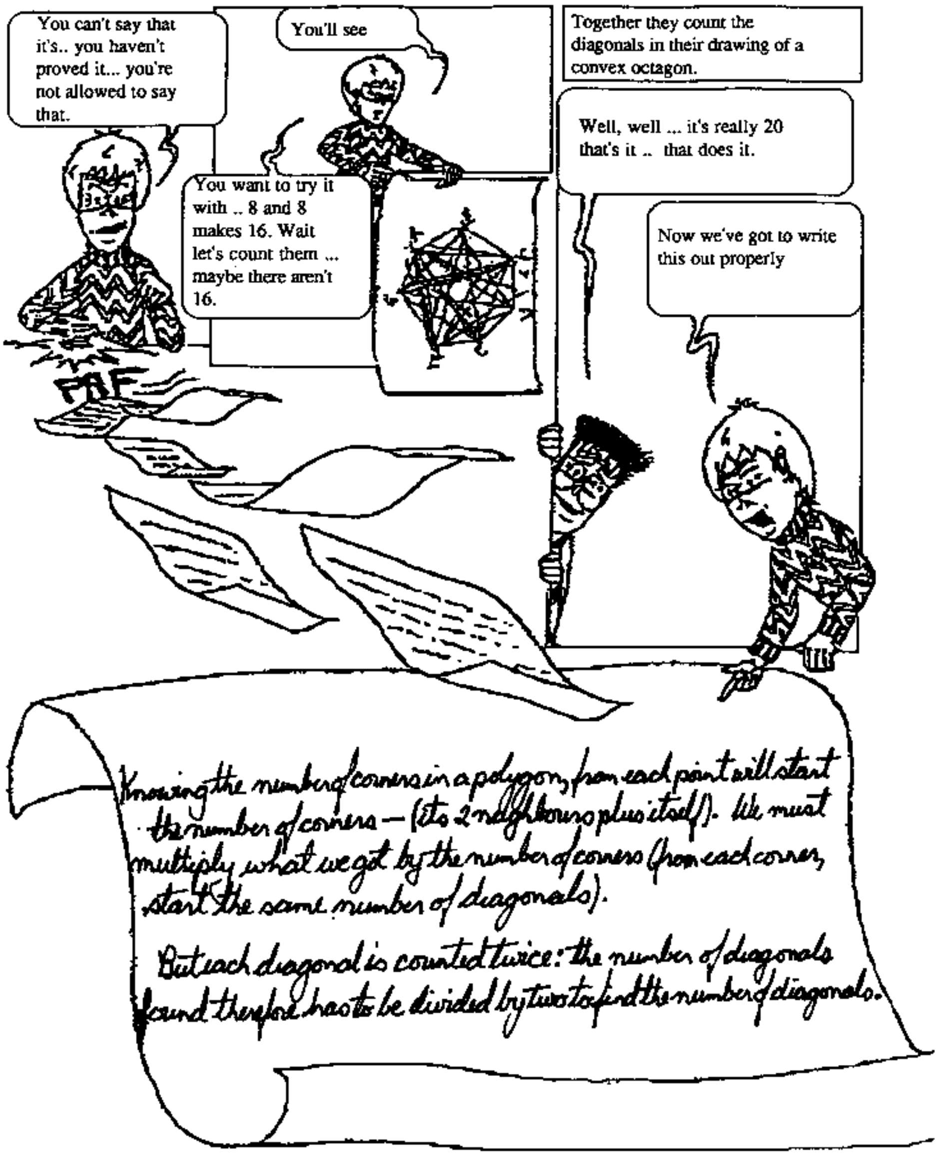
You may notice the juxtaposition of authoritative arguments next to arguments resting on the mathematical nature of and relationships among the objects in the problem-solving process. The very reasons for their final solution: if n is the number of vertices then $n(n-3)/2$ is the number of diagonals, are not used to produce a proof. Especially, facing Chris' naive empiricism, it is a crucial experiment that Bert uses to overcome their conflict. Nevertheless, those reasons appear in their verbal exchanges, but it is a long way from knowledge used in action, even explicitly, to possession of such knowledge as a tool to express a mathematical proof.

Appendix to Chapter 24


Bert and Chris go back to the 7 - sided polygon
If you have a number and divide it by 2, you get 14.
Wait. there are 7 ....
Wait , let's start again. There are 7 corners by ...take away 3 to find..
So you know...you get double.
No..no..look, 4.. 7 times 4, 28 ..divided by 2..that's 14.
That's what I said...it's 7 and 7, isn't it?
Not at all, look ...each diagonal appears twice and dividing by 2 will give each diagonal once.
That's a complicated trick. Some people won't get it.
Anyway you haven't proved it. You're not allowed to say what you said.
Wait! Let me say something..You multiply the number of corners by 2, so if you've got 7 corners, that's 14, so you know for certain that there are 14 diagonals.
But how do you know, you haven't proved it..
People aren't stupid they count the number of corners
But they don't want to draw it. They want to calculate it!
So what have we done? Didn't we draw something?
Yeah but they...but they don't want to draw... they want a rule, that's what they want
Ok tell me your trick. I don't get it ... you get each diagonal once.
This trick looks fishy to me.
OK... if you've got 8 - 3, that makes 5. It'll be 5 for each corner, so ... that's 5 times 8. That's 40, divided by 2, that's 20.
That'll be 20. Wait. Let's see if it really is.
'Cos 8 and 8 isn't 20 .. but it works with 7
Yeah but .. maybe it's you .. maybe it's luck that we started with 7 and it worked ... 7 and 7 ... 14, so ...
So that's not it. It isn't the right way.
Oh! You are stupid!


Come on take a polygon with 7 sides...uhh...each corner has got two neighbours .. and it's got....5 diagonals.
Hey..wait. Let's see. make one here...make one..
Yes you do...to calculate it.
No. You don't need to draw it
Bert draws all the diagonals from one vertex of the polygon that Chris has just drawn.
Only 4 diagonals! No...no...We forgot itself! 7 - 2 + itself. You can't draw a diagonal to itself!
So that's it..yeah!
Oh..no..no..do...you have to do the same for each corner
OK..wait..wait. We'd better check out what you said..yeah..'cos
Yeah..wait.. I'll explain.
Yeah...let's multiply what we got by the number of points, that will be the same thing as starting from each corner. That's it...yeah
Knowing the number of corners in a polygon, from each point will start the number of corners - (its two neighbours + itself). Multiply what we get by the number of corners.
So for 7 you get...you get 4...4 times 7! 'Cos there are 7 corners.
Yeah..so we're going to count them.
Uhhhh!
No! It isn't right. It was OK before.....but....since there are some that....some like AF and FA....Noooooo!
....Yeah...got it...... think I've got it...ummm....take away the number of corners from what you get...'cos...
'cos the diagonal AF is...it's the diagonal FA...BF....no it doesn't work....uhhh.


Chris counts again the diagonals of a seven-sided polygon
So that's 7 and 7 ...14!
14!
That'll do it...that's the number of sides of the polygon...double...go on..do it with 8 sides..you'll see.
Yeah, sure ..that'll be 16.
But we can't say that.. we haven't proved it we can't be sure it's right.
What...prove it all over again.
How do you know it's double.
Inside the brackets...if this thing has 7 corners..there's no need to prove it. See you don't need to ..you put..look.
Why double?...how's that?...you don't know what you're talking about.
Yeah I do know...that's it...that's what we're not allowed to put.
I know...I think I've got it. I think I've got it. Wait...give me the pen ....multiply...multiply by each corner to get ...wait .. you have to multiply what's here by the number of corners.
Oh so you're saying some could get counted 3 times.
But ...but ... there are the diagonals that get .. that would be counted twice ... and ... No! Every diagonal gets counted twice.
That's right!
No .. no not at all ...not at all.. AF has only been counted twice ... when you draw all the diagonals through A and all the diagonals through F .. and through B, it's all the same thing ..
OK, so it's twice then .. count them twice .. so that's it.
It's like dividing by 2.
Knowing the number of corners in a polygon, from each point will start, the number of corners - (its two neighbours + itself)
We have to multiply what we get by the number of corners (from each corner start the same number of diagonals).
BUT, each diagonal has been counted twice.
The number of diagonals has to be divided by two and we get each diagonal once.



(A number of people worked on translating this cartoon: see the Acknowledgements p. ix for details.)